# Technical Comments

## Comment on "Mean Velocity Profile of a Thick Turbulent Boundary Layer along a Circular Cylinder"

PETER BRADSHAW*
*Imperial College of Science and Technology, London, England*
AND
V. C. PATEL†
*University of Iowa, Iowa City, Iowa*

CHASE[1] shows that the most satisfactory of a number of formulae for the "law of the wall" in turbulent flow over a slender axisymmetric body of radius $a$ is that of Rao.[2,3] The purpose of this Note is to point out that the argument on which Rao's formula is based is unreliable, but that a more reliable argument gives numerical values very close to Rao's and therefore close to the experimental data.

Rao argues that because in the viscous sublayer on a slender body the usual linear law $u/v_* = v_* y/\nu$ is replaced by $u/v_* = (v_* a/\nu)\ln(1+y/a)$, the usual law of the wall that extends into the turbulent region,

$$u/v_* = f(v_* y/\nu) \tag{1}$$

should be replaced by

$$u/v_* = f[(v_* a/\nu)\ln(1+y/a)] \tag{2}$$

where the function $f$ is the same in both cases. In particular Rao replaces the usual logarithmic law

$$u/v_* = (1/K)\ln(v_* y/\nu) + C \tag{3}$$

by

$$u/v_* = (1/K)\ln\{(v_* y/\nu)[(a/y)\ln(1+y/a)]\} + C \tag{4}$$

Now Rao's argument should be equally valid for plane boundary layers with suction or injection at velocity $V_w$, where the linear law is replaced by

$$u/v_* = (v_*/V_w)[\exp(V_w y/\nu) - 1] \tag{5}$$

However, the law of the wall derived by replacing $v_* y/\nu$ in (3) by the right-hand side of (5) is distinctly different from the "bilogarithmic" law[4] which is generally accepted as a good fit to experimental data outside the viscous sublayer; for instance if $V_w/v_* = 0.1$ the value of $u/v_*$ at $v_* y/\nu = 100$ predicted by the present adaptation of the Rao argument is about 35, whereas the bilogarithmic law gives only 22.8, or less if the observed decrease of $C$ with $V_w/u_\tau$ is allowed for. Therefore the a posteriori justification of Rao's argument for axisymmetric flow provided by Chase is not repeated in the case of plane transpired flow. Since there is no a priori reason why the fully-turbulent flow should respond to transpiration or curvature in the same way as the sublayer, one must conclude that the success of Rao's formula for axisymmetric flow is largely a matter of luck.

Now the successful bilogarithmic law mentioned above follows from the so-called "mixing length formula"

$$\partial u/\partial y = (\tau/\rho)^{1/2}/Ky \tag{6}$$

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* Reader, Department of Aeronautics.
† Associate Professor, Institute of Hydraulic Research.

Table 1 Multiplying factors in the modified logarithmic laws

| $y/a$ | 1 | 2 | 3 | 10 |
|---|---|---|---|---|
| Rao, Eq. (4) | 0.694 | 0.546 | 0.462 | 0.240 |
| Mixing length, Eq. (7) | 0.690 | 0.535 | 0.444 | 0.215 |

where $\tau$ is the shear stress at height $y$. We proceed to show that (6) yields a law of the wall for slender axisymmetric bodies that is very close to Rao's modified logarithmic law, (4), and therefore close to the experimental data. Equation (6) is based on similarity arguments, discussed by Townsend[4]; the basic assumptions are that the eddy length scales are proportional to distance from the surface and that the eddy velocity scales are proportional to $(\tau/\rho)^{1/2}$. These assumptions are found to be valid for $y/\delta < 0.2$ in a plane boundary layer, but in a slender axisymmetric flow there will be an upper limit on $y/a$ as well as $y/\delta$; if $y/a$ is large, the lateral length scale of the eddies will be much larger than $a$, so the eddies will not be constrained as they would be by an effectively infinite wall and any law of the wall analysis, including Rao's, will be invalid. $y/a = 3$ is an optimistic figure for the upper limit.

Integrating (6) with $\tau = \tau_w/(1+y/a)$—the appropriate variation for an external axisymmetric flow which is changing slowly in the $x$ direction—and requiring compatibility with (3) for small $y/a$ gives

$$u/v_* = (1/K)\ln\{(v_* y/\nu)[4/(1+(1+y/a)^{1/2})^2]\} + C \tag{7}$$

Note that the "derivative hypothesis" mentioned by Chase would correspond to (6) with $\tau = \tau_w(1+y/a)^2$ so that for small $y/a$ its deviation from the planar law of the wall is about twice that of (7). The multiplying factors in square brackets in (4) and (7) both tend to $1 - y/2a$ for small $y/a$; values for larger $y/a$ are shown in Table 1, and the difference between Rao's formula and the mixing length formula in the practical range of $y/a$ is seen to be negligible. Equation (6) was used by Huffman and Bradshaw[5] to correlate velocity profiles in plane-wall flows and axisymmetric flows, both internal and external, and no consistent discrepancy could be found. It was, however, found that the constant of integration which represents the effect of the viscous sublayer, $C$ in (7), depended on the dimensionless shear stress gradient at the wall, $(\nu/\rho v_*^3)(\partial\tau/\partial y)_w$ or $\nu/v_* a$ for axisymmetric flow, if the latter was numerically larger than 0.001. Rao's scaling does not, of course, reproduce this effect: (6) does not purport to; an empirical variation of $C$ can be imposed on (4) or (7). A specific advantage of (6) is that it can be integrated with any shear-stress profile and thus be applied to accelerated or retarded flows.

The success of the conventional formula, (6), in matching the data analyzed by Chase implies that conventional differential methods of boundary-layer calculation—most of which reduce to (6) near the surface—should be valid near the surface in slender axisymmetric flow, and that special scaling of the kind suggested by Rao is not needed.

### References

[1] Chase, D. M., "Mean Velocity Profile of a Thick Turbulent Boundary Layer," *AIAA Journal*, Vol. 10, No. 7, July 1972, pp. 849–850.

[2] Rao, G. N. V., "The Law of the Wall in a Thick Axisymmetric Turbulent Boundary Layer," *Journal of Applied Mechanics*, Vol. 34, 1967, pp. 237–238.

[3] Rao, G. N. V. and Keshavan, N. R., "Axisymmetric Turbulent

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[4] Townsend, A. A., "Equilibrium Layers and Wall Turbulence," *Journal of Fluid Mechanics*, Vol. 11, 1961, pp. 97–116.

[5] Huffman, G. D. and Bradshaw, P., "A Note on Von Kármán's Constant in Low Reynolds Number Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 53, 1972, pp. 45–60.

# Reply by Author to P. Bradshaw and V. C. Patel

DAVID M. CHASE*
*Bolt Beranek and Newman Inc., Cambridge, Mass.*

BRADSHAW and Patel's comments are most cogent. The comparison of hypotheses for the mean velocity profile in turbulent axisymmetric flow along a cylinder given in Ref. 1, though correct, did not include the general hypothesis discussed by Bradshaw or, otherwise viewed, failed to define the most likely form of the "derivative" hypothesis.† It is this modified hypothesis cited by Bradshaw (hereafter called the local-similarity hypothesis), apart from inessential complications, that was applied to cylindrical flows in Refs. 2 and 3.

As defined in Ref. 1, the derivative hypothesis corresponded, in the logarithmic range, to a ratio of shear stress to eddy viscosity given by $(\tau/\rho)/\varepsilon = Av_*/y(1+y/a)$. Though, as stated by Bradshaw, this hypothesis corresponds to his Eq. (6) with $\tau = \tau_w/(1+y/a)^2$, it likewise corresponds—perhaps more sensibly though equally invalidly—to $\tau = \tau_w/(1+y/a)$ but with $\varepsilon = Ky(\tau_w/\rho)^{1/2}$ instead of $\varepsilon = Ky(\tau/\rho)^{1/2}$, i.e., with the eddy velocity scale assumed equal to its value at the wall instead of proportional to local shear-stress velocity. The local-similarity hypothesis, we note, implies an eddy viscosity given by $\varepsilon = A^{-1}v_*y(1+y/a)^{-1/2}$ in the logarithmic range, whereas Rao's hypothesis implies $\varepsilon = A^{-1}v_*a \ln(1+y/a)$.

Further quantitative comparison merits comment. The profile measured by Richmond[4] for $U = 460$ cm/sec with clay-centerbody trip and enveloping stovepipe, mentioned in Ref. 1, was especially smooth and the value of $av_*/v$ small. A comparison like that of Fig. 1, Ref. 1, of the measured and variously computed profiles is given for this instance in Fig. 1 below with $v_*$ adjusted in each computation to yield the measured result at a certain large $y$ and with the planar profile taken as in Ref. 1, Eq. (7). (A comparison like that of Fig. 2, Ref. 1, is omitted but readily envisioned.) Rao's hypothesis is seen to yield excellent agreement with the measured profile even out to $y/a \sim 50$. As stated by Bradshaw, we have no right to expect any law-of-the-wall analysis to retain validity at such large $y/a$, and, furthermore, the assumption of local similarity finds success more generally than the argument for Rao's hypothesis. If accidental, the accuracy of the Rao profile is indeed remarkable.

A profile may be computed for the example of Fig. 1 also from the local-similarity hypothesis by use of Bradshaw's Eq. (7) for the logarithmic range with parameters $v_*$ and $C$ both adjustable. No profile so computed conforms to the measured one in Fig. 1 over such a large range of $y/a$ (outside the sublayer) as does the Rao curve. Neither, however, does a Rao profile of the analogous approximate form

$$u/v_* = A \ln[(av_*/v)\ln(1+y/a)] + B$$

with $v_*$ and $B$ adjustable but without explicit treatment of the transitional profile. The Rao formulation retains the advantage of providing a simple explicit and successful prescription for incorporation of sublayer and transitional effects, e.g., by use of Eq. (7), Ref. 1.

## References

[1] Chase, D. M., "Mean Velocity Profile of a Thick Turbulent Boundary Layer along a Circular Cylinder," *AIAA Journal*, Vol. 10, No. 6, June 1972, pp. 848–850.

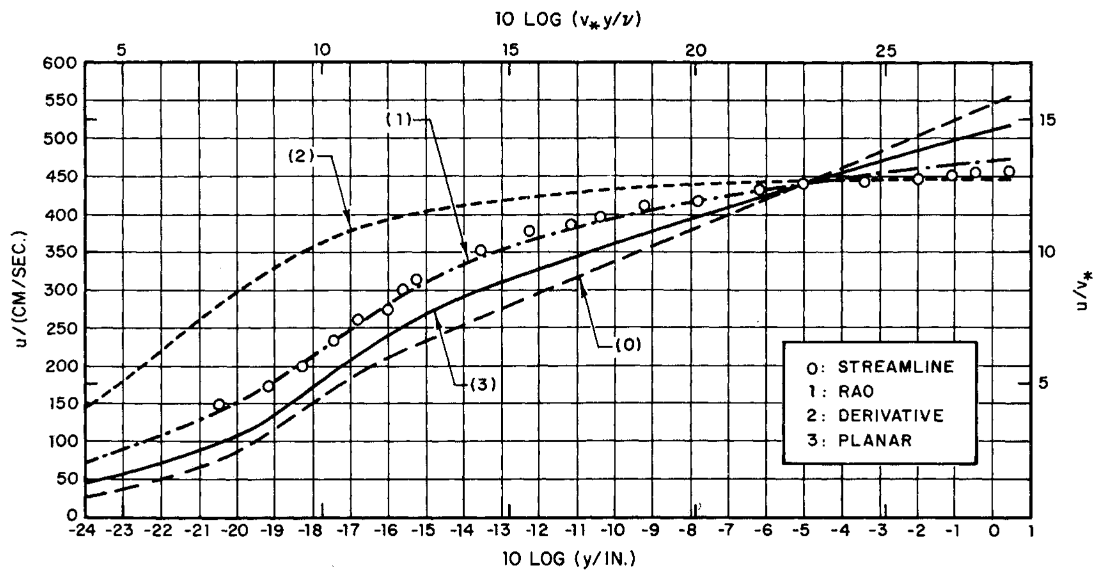


**Fig. 1 Measured mean velocity profile for cylinder[4] and those computed from various hypotheses[1] with wall-friction velocity chosen in each case to yield measured velocity at $y = 0.315$ in. Dimensionless scales refer to $v_* = 35.2$ cm/sec of Rao curve.**

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* Senior Scientist, Oceanology Department.

† The derivative hypothesis investigated in Ref. 1 was inadvertently stated incorrectly in Eq. (5). With $G(y_+, a_+) \equiv F(y_+, y_+/a_+)$, where $F$ was defined in Eq. (1), Eq. (5) should be replaced by

$$\partial G(y_+, a_+)/\partial y_+ = (1+y_+/a_+)^{-1}\, dF(y_+, 0)/dy_+$$

Bradshaw, however, made the intended interpretation.